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EQUATIONS DEFINING THE PERIODS OF TOTALLY DEGENERATE CURVES

BY

L. GERRITZEN

Fakultät und Institut für Mathematik, Ruhr-Universität Bochum
4630 Bochum 1, Postfach 102148, Gebaude NA 2/33, Germany

ABSTRACT

Mumford has studied the generalized Jacobian variety of a singular, irreducible curve in section 5 of his book (1984). It is determined by a period matrix which is a symmetric matrix whose diagonal is zero. The problem to determine systems of equations for the period matrices of totally degenerate curves is the analogue of the Schottky problem. An essentially complete solution is given.

Introduction

In this article an equation is derived which describes the locus of period matrices of stable totally degenerate curves of genus 4 whose intersection graph is simple. The equation is given by a polynomial F in the entries q_{ij} of a symmetric 4×4 matrix whose diagonal is zero.

This result allows one to deduce systems of equations for the periods of totally degenerate, irreducible curves of genus ≥ 5.

One can specialize F to obtain an equation $F' = 0$ describing the periods of totally degenerate, hyperelliptic curves of genus 3.

The irreducibility of Schottky's divisor in the space A_4 of principally polarized abelian varieties of genus 4 which is the locus of a modular form J was proved by Igusa, see [I]. In order to see how the function F can be obtained from J one has to degenerate J in some well-chosen toroidal compactification of A_4. The induction procedure in [vG] seems to degenerate to the simple one described in Section 5.

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In Section 1 the algebraic torus Q_g of totally degenerate symmetric $g \times g$ matrices is introduced.

In Section 2 the period map $\mathrm{per}_g : B^\star_{2g} \longrightarrow Q_g$ from the moduli scheme $B^\star_{2g}$ of $2g$-marked, stable trees of projective lines whose intersection graph has simple periods into Q_g is defined. The image P_g is the scheme of periods of totally degenerate curves, see [M], §5.

The equation $F' = 0$ for hyperelliptic 3×3 matrices is computed in Section 3. In Section 4 the equation $F = 0$ describing P_4 in Q_4 is deduced. In Section 5 equations defining P_g in the open subscheme $\{q_{ij} \neq 1\}$ of Q_g are given for $g \geq 5$.

1. The Torus of Totally Degenerate, Symmetric Matrices

1. Let M be a free $\mathbb{Z}$–module and let q be a symbol for a system of variables. Denote by $\mathbb{Z}[q, M]$ the group ring of M over $\mathbb{Z}$ for which the monomial in $\mathbb{Z}[q, M]$ associated to $m \in M$ is denoted by q^m.

Any $f \in \mathbb{Z}[q, M]$ is given by an expression $f = \Sigma_{m \in \nu} c_m q^m$ where ν is a finite subset of M and $c_m \in \mathbb{Z}$ for all m. Let $\chi_m : \mathbb{Z}[q, M] \longrightarrow \mathbb{Z}$ be the $\mathbb{Z}$–linear map which sends $q^{m'}$ onto

$$\delta_{mm'} := \begin{cases} 1 : m' = m, \\ 0 : m' \neq m. \end{cases}$$

Then $f = \Sigma_{m \in M} \chi_m(f) \cdot q^m$ for any $f \in \mathbb{Z}[q, M]$. It is called the expansion of f relative to the monomials (or characters) in $\mathbb{Z}[q, M]$.

The support $\mathrm{supp}(f)$ of f is defined to be $\{m \in M : \chi_m(f) \neq 0\}$. It is a finite subset of M. Let $\eta : M \longrightarrow \mathbb{Z}$ be a linear form on M, $\eta \not\equiv 0$, and $f \in \mathbb{Z}[q, M]$, $f \not\equiv 0$.

Definition: $\deg_\eta f := \sup\{\eta(m) : m \in \mathrm{supp} f\}$ is called the degree of f relative to η. $\zeta_\eta(f) := \sum \chi_m(f) \cdot q^m$ where the summation is over all $m \in \mathrm{supp} f$ for which $\eta(m) = \deg_\eta f$ is called the leading term of f relative to η.

One gets the following rules:

$$\deg_\eta f \cdot f' = \deg_\eta f + \deg_\eta f',$$
$$\zeta_\eta(f \cdot f') = \zeta_\eta(f) \cdot \zeta_\eta(f').$$

Proof: Both follow readily from the obvious formula

$$\chi_m(f \cdot f') = \sum_{\substack{n+n'=m \\ n,n' \in M}} \chi_n(f) \chi_{n'}(f'). \qquad \blacksquare$$

The group of units $\mathbb{Z}[q,M]^\star$ of $\mathbb{Z}[q,M]$ are those functions $f \not\equiv 0$ for which $\operatorname{supp}(f)$ consists of just one element m such that $\chi_m(f) \in \{+1,-1\}$; $f = \pm q^m$.

Definition: $\operatorname{ht}_\eta f := \deg_\eta f - \deg_{(-\eta)} f$ is called the height of f relative to η. Obviously $\operatorname{ht}_\eta(\pm q^m f) = \operatorname{ht}_\eta f$ and $\operatorname{ht}_\eta(ff') = \operatorname{ht}_\eta f + \operatorname{ht}_\eta f'$.

The convex hull of $\operatorname{supp}(f)$ in $\mathbb{R} \otimes M$ is denoted by $\operatorname{conv}(f)$.

2. Let $N = N_g$ be a free abelian group of rank $g, g \geq 2$, and $e_1, \ldots, e_g$ a base of N_g. Then the quotient

$$N_g' = N_g \otimes_{\text{sym}} N_g / \bigoplus_{i=1}^{g} \mathbb{Z} e_i^2$$

of the symmetric tensor product of N_g with itself by the subgroup generated by the squares $e_1^2 = e_1 \cdot e_1, \ldots, e_g^2 = e_g \cdot e_g$ is also a free abelian group. Its rank is

$$\binom{g}{2} = \frac{1}{2} g(g-1).$$

Let $Q_g := \operatorname{Spec}\mathbb{Z}[q, N_g']$. It is considered as an algebraic torus over $\mathbb{Z}$. It is canonically isomorphic to $\mathbb{G}_m^{\binom{g}{2}}$ where $\mathbb{G}_m$ denotes the multiplicative group scheme over $\mathbb{Z}$ given through the characters $q_{ij} = q^{e_i e_j}, 1 \leq i < j \leq g$.

Let k be a field. A k-valued point v of Q_g is a symmetric $g \times g$ matrix with entries in $k, v = (v_{ij})$, such that $v_{ii} = 0$ for all i and $v_{ij} \in k^\star = k - \{0\}$ for all $i \neq j$.

A matrix with these properties is called a **totally degenerate, symmetric matrix over** k. Q_g is called the **torus of totally degenerate, symmetric** $g \times g$ **matrices.**

3. Let Γ_g be the group of all automorphisms $\gamma : N_g \to N_g$ that map the set $\{\pm e_1, \ldots, \pm e_g\}$ onto itself. Γ_g is the group of all proper and improper movements of the standard g-dimensional cube $\{x \in \mathbb{R}^g : x = (x_1, \ldots, x_g),\ |x_i| \leq 1 \text{ for all } i\}$ in euclidean space $\mathbb{R}^g$. Γ_g is referred to as the moduli group on the torus of totally degenerate, symmetric $g \times g$ matrices.

Let $\bar{\Gamma}_g$ be the subgroup of Γ_g of all automorphisms which permute the set $\{e_1, \ldots, e_g\}$ for all i. Let Γ_g^0 be the subgroup of Γ_g of all automorphisms which permute the set $\{e_i, -e_i\}$ for all i. Then Γ_g^0 is a normal subgroup of Γ_g and Γ/Γ_g^0 is canonically isomorphic to $\bar{\Gamma}_g$. Γ_g is the semidirect product of Γ_g^0 with $\bar{\Gamma}_g$.

Let $\epsilon_i : N \longrightarrow N$ be that automorphism for which

$$\epsilon_i(e_j) = \begin{cases} e_j & : j \neq i, \\ -e_i & : j = i. \end{cases}$$

Then Γ_g^0 is generated by $\epsilon_1, \ldots, \epsilon_g$ and $\Gamma_g^0 \cong (Z/2\mathbb{Z})^g$

Γ_g acts canonically on N'_g. If $e_i e_j$ is the symmetric product of e_i with e_j, then

$$\gamma(e_i e_j) = (\gamma e_i) \cdot (\gamma e_j)$$

for any $\gamma \in \Gamma_g$.

Γ_g also acts canonically on the group ring $\mathbb{Z}[q, N'_g]$. If $m \in N'_g$, then $\gamma \cdot q^m = q^{\gamma(m)}$ and $\gamma \cdot f = \sum \chi_m(f) \cdot q^{\gamma(m)}$ for $f \in \mathbb{Z}[q, N'_g]$.

Γ_g acts on Q_g canonically from right. If $v = (v_{ij})$ is a k-valued point of Q_g, and if $\gamma \in \bar{\Gamma}_g$, then $v \cdot \gamma = w = (w_{ij})$ with $w_{ij} = v_{\gamma(i)\gamma(j)}$ where $\gamma(i) = l$ if $\gamma(e_i) = e_l$.

If $\epsilon_l \in \Gamma_g^0$, then $v \cdot \epsilon_l = w$ with

$$w_{ij} = \begin{cases} v_{ij}^{-1} & : i = l \quad \text{or} \quad j = l \\ v_{ij} & : \text{otherwise} \end{cases}$$

for $i \neq j$. Then $(\gamma f)(v) = f(v\gamma)$ for any $f \in \mathbb{Z}[q, N'_g]$ and any k-valued point v of Q_g.

4. In this section we take $g = 3$. Let

$$\begin{aligned} \Delta' &:= (q_{12} - 1)(q_{13} - 1)(q_{23} - 1) \\ &= q_{12}q_{13}q_{23} - q_{12}q_{13} - q_{12}q_{23} - q_{13}q_{23} + q_{12} + q_{13} + q_{23} - 1. \end{aligned}$$

Let $H' := q_{12}q_{13}q_{23} + q_{12} + q_{13} + q_{23}$. Let

$$G' := q_{12}(q_{13} - 1)^2(q_{23} - 1)^2 + q_{13}(q_{12} - 1)^2(q_{23} - 1)^2 + q_{23}(q_{12} - 1)^2(q_{13} - 1)^2$$

and $F' := \Delta' H' + G'$.

If M is any Γ_3-orbit in N'_3 we denote by S_M the Γ_3- invariant function given by $S_M := \Sigma_{m \in M} q^m$.

Let W' be the unit cube in $\mathbb{Q} \otimes \mathbb{N}'_3$, i.e.

$$W' = \left\{ \sum_{i<j} m_{ij} e_i e_j : -1 \leq m_{ij} \leq +1 \right\}.$$

LEMMA 1.4.1: *$W' \cap N_3'$ consist of the following Γ_3-orbits: $T_0 = \{0\}, T_1 = \Gamma_3 e_{12}, T_2 = \Gamma_3(e_{12}+e_{13}), T_3 = \Gamma_3 e, T_3' = \Gamma_3(-e)$ where $e := e_{12}+e_{13}+e_{23}$.*

We leave the proof to the reader.

Let $S_i := S_{T_i}$ Then

$$S_0 := 1,$$
$$S_1 = q_{12} + q_{13} + q_{23} + q_{12}^{-1} + q_{13}^{-1} + q_{23}^{-1},$$
$$S_3 = q_{12}q_{13}q_{23} + q_{12}^{-1}q_{13}^{-1}q_{23} + q_{12}^{-1}q_{13}q_{23}^{-1} + q_{12}q_{13}^{-1}q_{23}^{-1}.$$

PROPOSITION 1.4.2:

(i) *$\operatorname{conv}(q^{-e}F')$ is the tetrahedron T spanned by*

$$T_3 = \{e, -e_{12} - e_{13} + e_{23}, -e_{12} + e_{13} - e_{23}, e_{12} - e_{13} - e_{23}\}$$

which consists of the "even" corners of W'. T_1 consists of the centers of the edges of T while T_0 is the center of T.

(ii) *$q^{-e}F' = S_3 - 2S_1 + 8S_0$ and $q^{-e}F'$ is Γ_3-invariant.*

Proof: If $m = 2e_{12} + 2e_{13} + e_{23}$ one finds that $\chi_m(G') = 1$ and $\chi_m(\Delta' H') = -1$ and thus $\chi_{m-e}(q^{-e}F') = 0$. Also $\chi_{2e}(G') = 0$ and $\chi_{2e}(\Delta' H') = 1$ and thus $\chi_e(q^{-e}F') = 1$.

If $m = 2e_{12} + e_{13} + e_{23}$ one finds that $\chi_m(G') = -4$ and $\chi_m(\Delta' H') = 2$ and thus

$$\chi_{e_{12}}(q^{-e}F') = -2.$$

Also $\chi_e(G') = 3 \cdot 4$ and $\chi_e(\Delta' H') = -4$ and thus $\chi_0(q^{-e}F') = 8$. Furthermore $\chi_0(G') = \chi_0(\Delta' H') = 0$ and thus $\chi_{-e}(q^{-e}F') = 0$. This proves (i) and (ii).

Let k be a field of characteristic p.

PROPOSITION 1.4.3: *If $p = 2$, then $1 \otimes F' = (q_{12}q_{13}q_{23} + q_{12} + q_{13} + q_{23})^2$ in $k \otimes \mathbb{Z}[N_3']$. If $p \neq 2$, then $1 \otimes F'$ is irreducible. F' is irreducible in $\mathbb{Z}[N_3']$.*

This is achieved by simple computations. Also we note that F' vanishes at the matrix

$$\eta = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

because $S_M(\eta) = \sharp M$ and $\sharp T_3 = 4, \sharp T_1 = 6, \sharp T_0 = 1$ and therefore

$$F'(\eta) = \sharp T_3 - 2\sharp T_1 + 8 = 0.$$

The polynomial H' can be expressed as a theta function, in fact:

$$H' = q_{12}\Theta_2\left(q'^2; -\frac{q_{13}}{q_{12}}, -\frac{q_{23}}{q_{12}}\right)$$

where $\Theta_2(q,x) = 1 - x_1 - x_2 + q_{12}x_1x_2$ and where

$$q' = \begin{pmatrix} 0 & q_{12} \\ q_{12} & 0 \end{pmatrix}, \quad q'^2 = \begin{pmatrix} 0 & q_{12}^2 \\ q_{12}^2 & 0 \end{pmatrix}.$$

5. In this section we take $g = 4$. Let

$$\Delta := \prod_{1\le i<j\le 4} (q_{ij} - 1) = (q_{12}-1)(q_{13}-1)(q_{23}-1)\cdot(q_{14}-1)(q_{24}-1)(q_{34}-1).$$

Let

$$\begin{aligned} H :=& q_{12}q_{13}q_{14}q_{23}q_{24}q_{34}q_{12}q_{14}q_{24} - q_{13}q_{14}q_{34} - q_{23}q_{24}q_{34} \\ &- q_{12}q_{13}q_{23} + q_{23}q_{14} + q_{13}q_{24} + q_{12}q_{34}. \end{aligned}$$

Let

$$\begin{aligned} G :=& q_{12}q_{34}(q_{13}-1)^2(q_{14}-1)^2(q_{23}-1)^2(q_{24}-1)^2 \\ &+ q_{13}q_{24}(q_{12}-1)^2(q_{14}-1)^2(q_{23}-1)^2(q_{34}-1)^2 \\ &+ q_{14}q_{23}(q_{12}-1)^2(q_{13}-1)^2(q_{24}-1)^2(q_{34}-1)^2 \end{aligned}$$

and $F := \Delta H - G$.

PROPOSITION 1.5.1: *$q^{-e}\Delta H, q^{-e}G$ and $q^{-e}F$ are Γ_4-invariant where*

$$e := \sum_{i<j} e_{ij} = e_{12} + e_{13} + e_{14} + e_{23} + e_{24} + e_{34}.$$

Proof: If $\gamma \in \bar\Gamma_4$ is a permutation of e_1, e_2, e_3, e_4, then $\gamma\Delta = \Delta$, $\gamma H = H$, $\gamma G = G$. If

$$\gamma = \epsilon_1, \qquad \epsilon_1 e_i := \begin{cases} e_i & : i > 1, \\ -e_1 & : i = 1, \end{cases}$$

then

$$\epsilon_1\Delta = (-1)\frac{1}{q_{12}q_{13}q_{14}}\cdot\Delta,$$
$$\epsilon_1 H = (-1)\frac{1}{q_{12}q_{13}q_{14}}\cdot H,$$
$$\epsilon_1 G = \frac{1}{q_{12}^2q_{13}^2q_{14}^2}\cdot G.$$

One concludes that

$$\epsilon_1 q^{-e}\Delta H = q^{-e}\Delta H,$$
$$\epsilon_1 q^{-e} G = q^{-e} G.$$

As $\gamma\epsilon_1\gamma^{-1} = \epsilon_i$ for some $\gamma \in \Gamma_g$ we are through with the proof. ∎

One can express H as a theta function. If

$$\Theta_3(q,x) = 1 - x_1 - x_2 - x_3 + q_{12}x_1x_2 + q_{13}x_1x_3 + q_{23}x_2x_3 - q_{12}q_{13}q_{23}x_1x_2x_3,$$

one gets:

$$H = -(q_{12}q_{13}q_{23})\Theta_3\left(q'^2, \frac{x_1}{q_{12}q_{13}}, \frac{x_2}{q_{12}q_{23}}, \frac{x_3}{q_{13}q_{23}}\right)$$

where

$$q' = \begin{pmatrix} 0 & q_{12} & q_{13} \\ q_{12} & 0 & q_{23} \\ q_{13} & q_{23} & 0 \end{pmatrix}, \quad q'^2 := (q_{ij}^2)$$

and $x_i = q_{4i}$.

Denote by $f|_{q_{i4}=-1}$ the function obtained from $f \in \mathbb{Z}[N_4']$ by substituting (-1) for q_{14}, q_{24} and q_{34}. Then

$$\Delta|_{q_{i4}=-1} = (-1)\cdot 2^3\Delta',$$
$$H|_{q_{i4}=-1} = (-2)H',$$
$$G|_{q_{i4}=-1} = (-1)2^4 G',$$
$$F|_{q_{i4}=-1} = 2^4\cdot F'.$$

PROPOSITION 1.5.2: *F is irreducible in $\mathbb{Z}[N_4']$.*

Proof: Assume that F is reducible in $\mathbb{Z}[N_4']$. As $\chi_{2e}(F) = 1$, F is primitive and thus each prime factor of F is also primitive.

There must exist a prime factor f of F such that $f|_{q_{i4}=-1} = 2^r\cdot F'$, $0 \le r \le 4$. But then $\mathrm{ht}_{\eta_{ij}} f = 2$ for all $i < j$ which shows that $f = \pm q^m F$.

2. Periods of Totally Degenerate Curves

1. Let C be a stable totally degenerate curve of genus $g \geq 2$ over a field k. One can find stable, $2g$–marked trees of projective lines $X = (\tilde{C}, a, b)$ such that $C = \tilde{C} \bmod a = b$ where $a = (a_1, \ldots, a_g)$, $b = (b_1, \ldots, b_g)$ are g–tupels of k–rational points on $\tilde{C}$, see [GHP], [G]. There is a canonical base $\omega_1, \ldots, \omega_g$ of differentials of first kind on C such that $\mathrm{res}_{a_i}\omega_i = +1$, $\mathrm{res}_{b_i}\omega_i = -1$, ω_i without poles on $C - [a_i, b_i]$, where $[a_i, b_i]$ denote the set of double points in $\tilde{C}$ between a_i and b_i, see [G], §1 also [H], Chap IV, §1, Exerc. 1.9, p. 298.

Let

$$E_X := \{u = (u_1, \ldots, u_g),\ u_i \text{ rational function on } C \text{ such that } \frac{du_i}{u_i} = \omega_i\}.$$

Let per u be the $g \times g$ matrix $\eta = (\eta_{ij})$ such that

$$\eta_{ij} = \frac{u_i(a_j)}{u_i(b_j)} \quad \text{for } i \neq j, \quad \eta_{ii} = 0.$$

Let $G(C)$ be the intersection graph of C.

Definition: $G(C)$ is simple, if two simple closed unoriented paths γ, γ' in $G(C)$, $\gamma \neq \gamma'$, have no common edge.

PROPOSITION 2.1.1: *Assume that $G(C)$ is simple. Then the following holds:*

(i) *For any $u \in E_X$ the matrix* per u *is symmetric and its entries outside the diagonal are in $k^\star$.*

(ii) per u = per u' *for any* $u, u' \in E_X$

Proof: Let $i \neq j$. There is a unique component Y of X such that $\pi_Y(a_j) \neq \pi_Y(a_i)$, $\pi_Y(a_j) \neq \pi_Y(b_i)$ where π_Y denotes the projection from X onto Y. As the interval from a_i to b_i in X has no common double point with the interval from a_j to b_j, also $\pi_Y(b_j) \neq \pi_Y(a_i), \pi_Y(b_j) \neq \pi_Y(b_i)$. Then $u_i(a_j)/u_i(b_j)$ is the cross–ratio of the points $\pi_Y(a_i)$, $\pi_Y(b_i)$, $\pi_Y(b_j)$, $\pi_Y(a_j)$ which is an element in $k^\star$. This shows (i) and (ii).

Definition: per X := per u for $u \in E_X$ if $G(C)$ is simple. per X is called the period matrix of X; it is a k–valued point of Q_g.

2. Let B_{2g} be the fine moduli scheme of stable $2g$–marked trees of projective interval, see [GHP]. Let $B_{2g}^\star$ be the open subscheme of B_{2g} which consists of the simple $2g$–marked trees which shall mean that the graph obtained from the tree by adding edges between a_i and b_i is simple. $B_{2g}^\star$ is open because it is the complement of the union of all hypersurfaces $\{\lambda_{i,g+i,g+j,j} = 0\}$ where λ_ν denotes the canonical variable associated to the quadruple $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$, $\nu_i \in \{1, \ldots, 2g\}$ on B_{2g}, see [GHP], §1.

There is a morphism $\mathrm{per}_g : B_{2g}^\star \longrightarrow Q_g$ with the following property: if X is a k–valued point of $B_{2g}^\star$, it is a stable $2g$–marked tree of projective lines, then $\mathrm{per}_g(X) = \mathrm{per}X \in Q_g(k)$.

PROPOSITION 2.2.1: *$\mathrm{per}_g(B_{2g}^\star)$ is a Zariski–closed subscheme of Q_g and per_g is proper.*

Proof: Let $M := \{(i,j) : 1 \leq i < j \leq g\}$. One embeds Q_g into the M–fold product $\mathbb{P}_1^M$ by means of the variables $q_{ij}, i < j$. Then $\mathbb{P}_1^M$ is a torus embedding of Q_g The morphism per_g can be extended to a morphism $\widehat{\mathrm{per}}_g : B_{2g} \longrightarrow \mathbb{P}_1^M$ sending a point X in B_{2g} onto the matrix $\eta = (\eta_{ij})$, $\eta_{ij} := \lambda_{i,-i,-j,j}(X)$.

Thus $\widehat{\mathrm{per}}_g(B_{2g})$ is a closed subscheme of $\mathbb{P}_1^M$ as B_{2g} is a projective scheme over $\mathbb{Z}$. One can easily check that a point of $\widehat{\mathrm{per}}_g(B_{2g})$ is in $Q_g \subseteq \mathbb{P}_1^M$ if and only if X is a simple $2g$–marked tree.

Thus $\widehat{\mathrm{per}}_g(B_{2g}) \cap Q_g = \mathrm{per}_g(B_{2g}^\star)$.

Definition: $P_g := \mathrm{per}_g(B_{2g}^\star)$. It is called the scheme of period matrices of totally degenerate curves with simple intersection graph.

The question to characterize P_g within Q_g is the analogue of the Schottky problem for totally degenerate curves, see e.g. [F], [MF], appendix to Chap. 7. It is hoped that a characterization of P_g will shed some light on the Schottky problem.

Of course $P_g = Q_g$ for $g \leq 3$. But P_4 is a hypersurface in Q_4

3. Γ_g acts on B_{2g} from the right. If $X = (Y, a, b)$ is a k–valued point of $B_{2g}, \gamma \in \Gamma_g$, then $X\gamma = (Y, a^\gamma, b^\gamma)$ and a^γ, b^γ are defined as follows: if $\gamma(e_i) = e_j$, then $a_i^\gamma := a_j$ and $b_i^\gamma := b_j$, and if $\gamma(e_i) = -e_j$, then $a_i^\gamma := b_j$ and $b_i^\gamma := a_j$.

Γ_g acts also on the open subscheme $B_{2g}^\star$ and the period map $\mathrm{per}_g : B_{2g}^\star \longrightarrow Q_g$ is Γ_g–equivariant.

Let

$$\epsilon = \epsilon_1 \epsilon_2 \cdot \ldots \cdot \epsilon_g, \ \epsilon_i(e_j) := \begin{cases} e_j & : j \neq i, \\ -e_i & : j = i. \end{cases}$$

Then ϵ acts trivially on Q_g while it acts non-trivially on $B_{2g}^\star$ if $g \geq 3$. Thus per_g induces a morphism $\overline{\mathrm{per}}_g$ from the quotient $B_{2g}^\star / < \epsilon >$ into Q_g which is called the **reduced period map**.

Let $X = (Y, a, b)$ be a $2g$–marked, stable tree of projective lines. Let $C(X) = Y \bmod a = b$ be the stable curve obtained from Y by identifying a_i with b_i for all i. Then $C(X)$ is a stable curve of genus g which is called the **curve associated to** X.

There is a reduced scheme D_g which is the categorial quotient of B_{2g} modulo the equivalence relation: $X \sim X'$ iff $C(X)$ is isomorphic to $C(X')$.

D_g is the coarse reduced moduli scheme of totally degenerate stable curves of genus g. There is a surjective morphism $\pi : B_{2g}/\Gamma_g \longrightarrow D_g$. The fiber of π over a curve $C \in D_g(k)$ is parametrized by the maximal subtrees of the graph $G(C)$ of C.

By considerations as in the proof of Proposition (2.1.1), one can show:

PROPOSITION 2.3.1: per_g *induces a morphism* $\mathrm{tor}_g : D_g \longrightarrow Q_g/\Gamma_g$ *which is called the Torelli map for totally degenerate, stable curves of genus* g.

The scheme Q_g/Γ_g is the analogue to the moduli space of principally polarized abelian varieties of genus g.

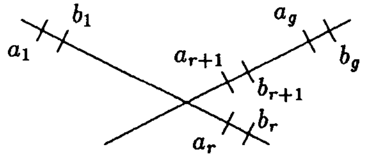
4. Let $\nu \subset \{1, 2, \ldots, 2g\}, \sharp\nu \geq 2$, and $B_{2g}(\nu)$ the closed subscheme of B_{2g} which contains all the $2g$–marked trees of projective lines with two components X_1, X_2 such that $a_i \in X_1$ for all $i \in \nu$ while $a_i \in X_2$ for all $i \notin \nu$. Here $a_i := b_{i-g}$ if $i \geq g+1$.

Then an open subscheme of $B_{2g}(\nu)$ is in $B_{2g}^\star$ iff one of the following cases occurs:

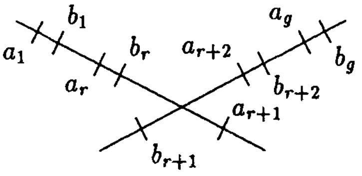
(i) there is a subset $\nu' \subset \{1, \ldots, g\}$ such that $\nu = \nu' \cup \{g + i : i \in \nu'\}$,

(ii) there is a subset $\nu' \subset \{1, \ldots, g\}$ and an element $k \in \{1, \ldots, 2g\} - \nu', k \notin \nu', k - g \notin \nu'$ such that $\nu = \nu' \cup \{g + i : i \in \nu'\} \cup \{k\}$.

If $\nu' = \{1, \ldots, r\}$, then in case (i) the generic $2g$-marked curve looks as follows:



If $\nu' = \{1, \ldots, r\}$, $k = r + 1$, then in case (ii) the generic $2g$-marked curve looks as follows:



PROPOSITION 2.4.1: *Let X be a k-valued point of $B_{2g}(\nu) \cap B^{\star}_{2g}$ and assume that ν is of type* (i). *Then if $\eta = \mathrm{per}X$ one has*

$$\eta_{ij} = 1$$

if $i \in \nu'$, $j \in \{1, \ldots, g\} - \nu'$. If X has two components and if $g \geq 3$, the fiber of the period map per_g over X is 1-dimensional if r or $g - r$ is equal to 2. Otherwise $\dim \mathrm{per}_g^{-1}(X) = 2$.

Remark: If $\nu' = \{1, \ldots, r\}$ then per X is of the block form

$$\begin{pmatrix} \eta' & 1 \\ 1 & \eta'' \end{pmatrix},$$

η' is a $r \times r$ matrix, η'' is a $(g-r) \times (g-r)$ matrix.

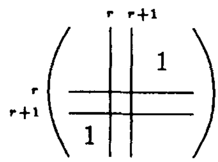
PROPOSITION 2.4.2: *Let X be a k-valued point of $B_{2g}(\nu) \cap B^{\star}_{2g}$ and assume that ν is of type* (ii).

For $\eta = \mathrm{per}X$ one has

$$\eta_{ij} = 1$$

if $i \in \nu'$, $j \in \{1, \ldots, g\} - (\nu' \cup \{k\})$.

Remark: If $\nu' = \{1, \ldots, r\}$, $k = r+1$, then per$X$ has the form



where the 1 in the upper half denotes a $r \times (g-r-1)$ matrix. The proofs of (2.4.1) and (2.4.2) are left to the reader.

5. Let B_{2g}^{hyp} be the closed subscheme of B_{2g} of fixed points of the automorphism on B_{2g} given by the action of $\epsilon = \epsilon_1 \epsilon_2 \cdots \epsilon_g$ on B_{2g}.

A k-valued point X of B_{2g} belongs to B_{2g}^{hyp} iff there is an automorphism σ_X of the tree of projective lines underlying X which maps a_i to b_i and b_i to a_i for all i. Then $\sigma_X \circ \sigma_X = \text{id}$ and σ_X is uniquely determined by X. If X is irreducible, $X = (\mathbb{P}_1 \times k,\ a, b)$ and if z is global coordinate on $\mathbb{P}_1 \times k$ such that

$$z(a_i) = x_i,\ z(b_i) = y_i,\ x_1 = 0,\ y_1 = \infty$$

then X is hyperelliptic (i.e. $X \in B_{2g}^{\text{hyp}}$) iff $x_i y_i = x_j y_j$ for all $i, j \geq 2$ and σ_X is the involution $z \longmapsto 1/z$.

Proof: Let $X^\epsilon = (\mathbb{P}_1 \times k,\ a', b')$. If $z(a'_i) = x'_i$, $z(b'_i) = y'_i$, $x'_1 = 0$, $y'_1 = \infty$, then $x'_i = 1/y_i$, $y'_i = 1/x_i$. Thus X^ϵ is isomorphic to X iff there is $\lambda \in k^*$ such that $(\lambda x'_i,\ \lambda y'_i) = (x_i, y_i)$ for all $i \geq 2$. This is the case iff $x_i y_i = \lambda$ for all $i \geq 2$. ∎

Let $P_g^{\text{hyp}} := \text{per}_g(B_{2g}^{\text{hyp}} \cap B_{2g}^*)$. It is a closed subscheme of Q_g. While $P_2^{\text{hyp}} = Q_2$, already P_3^{hyp} is a hypersurface in Q_3.

6. Let $X = (\mathbb{P}_1 \times k, a, b)$ be a $2g$-marked, stable projective line over a field k. Introduce a global coordinate z on $\mathbb{P}_1 \times k$ such that $z(a_1) = 0$, $z(b_1) = \infty$, $z(a_2) = 1$ and put $x_i := z(a_i)$, $y_i = z(b_i) \in k$ for $i \geq 2$. Then $\text{per}_g X$ is the $g \times g$ matrix $v = (v_{ij})$ such that

$$v_{1i} = \frac{x_i}{y_i} \quad \text{for } i \geq 2,$$

$$v_{ij} = \frac{\left(\frac{x_i - x_j}{x_i - y_j}\right)}{\left(\frac{y_i - x_j}{y_i - y_j}\right)} = \frac{(x_i - x_j)(y_i - y_j)}{(x_i - y_j)(y_i - x_j)} \quad \text{for } 2 \leq i < j \leq g.$$

Let

$$t_{ij} := \frac{x_i}{y_j} + \frac{x_j}{y_i} \quad \text{for } 2 \le i < j \le g.$$

Then

$$t_{ij} = v_{1j}\frac{x_i}{x_j} + v_{1i}\frac{x_j}{x_i}.$$

LEMMA 2.6.1: *For $2 \le i < j \le g$ one has*

$$t_{ij} = \frac{(v_{1i}v_{1j}+1)v_{ij} - (v_{1i}+v_{1j})}{(v_{ij}-1)}.$$

Proof:

$$\begin{aligned} v_{ij} &= \frac{(x_iy_i + x_jy_j) - (x_iy_j + x_jy_i)}{(x_iy_i + x_jy_j) - (x_ix_j + y_iy_j)} \\ &= \frac{\left(\frac{x_i}{y_j} + \frac{x_j}{y_i}\right) - \left(\frac{x_i}{y_i} + \frac{x_j}{y_j}\right)}{\left(\frac{x_i}{y_j} + \frac{x_j}{y_i}\right) - \left(\frac{x_ix_j}{y_iy_j} + 1\right)} \\ &= \frac{t_{ij} - (v_{1i} + v_{1j})}{t_{ij} - (v_{1i}v_{1j} + 1)} \end{aligned}$$

which shows the lemma. ∎

3. Periods of Hyperelliptic Curves of Genus 3

1. Let $X = (\mathbb{P}_1 \times k, a, b)$ be a 6–marked, stable, hyperelliptic projective line over k. We use the notation of (2.6) to obtain

$$t_{23} = \frac{x_2}{y_3} + \frac{x_3}{y_2} = \frac{1}{y_3} + \frac{1}{y_3} = \frac{2}{y_3}$$

as $x_3y_3 = x_2y_2 = y_2$. As $v_{12} = 1/y_2$, $v_{13} = x_3/y_3 = y_2/y_3^2$ one gets

$$4v_{12}v_{13} = t_{23}^2.$$

As

$$t_{23} = \frac{(v_{12}v_{13}+1)v_{23} - (v_{12}+v_{13})}{(v_{23}-1)}$$

one has the equation

$$4v_{12}v_{13}(v_{23}-1)^2 = A^2$$

with

$$\begin{aligned} A &= (v_{12}v_{13}+1)v_{23} - (v_{12}+v_{13}) \\ &= (v_{12}v_{13}+1)(v_{23}-1) + (v_{12}-1)(v_{13}-1). \end{aligned}$$

Let $\sigma = (v_{12}-1)(v_{13}-1)(v_{23}-1)$. Then

$$(v_{23}-1)^2 \cdot [(v_{12}v_{13}+1)^2 - 4v_{12}v_{13}] + 2\sigma(v_{12}v_{13}+1) + (v_{12}-1)^2(v_{13}-1)^2 = 0.$$

One substracts

$$g := v_{12}(v_{13}-1)^2(v_{23}-1)^2 + v_{23}(v_{12}-1)^2(v_{23}-1)^2 + v_{23}(v_{12}-1)^2(v_{13}-1)^2$$

on both sides to get

$$(-g) = (v_{23}-1)^2 B + 2\sigma(v_{12}v_{13}+1) + (v_{12}-1)^2(v_{13}-1)^2 - v_{23}(v_{12}-1)^2(v_{13}-1)^2$$

with

$$\begin{aligned}
B &= (v_{12}v_{13}+1)^2 - 4v_{12}v_{13} - v_{12}(v_{13}-1)^2 - v_{13}(v_{12}-1),\\
B &= (v_{12}v_{13}-1)^2 - (v_{13}-1)^2 - (v_{12}-1)(v_{13}-1)^2 - v_{13}(v_{12}-1)^2,\\
B &= v_{13}^2(v_{12}^2-1) - 2v_{13}(v_{12}-1) - (v_{12}-1)[(v_{13}-1)^2 + v_{13}(v_{12}-1)],\\
B &= (v_{12}-1)\cdot B',\\
B' &= v_{13}^2(v_{12}+1) - 2v_{13} - [(v_{13}-1)^2 + v_{13}(v_{12}-1)],\\
B' &= v_{13}^2 v_{12} + v_{13}^2 - 2v_{13} - v_{13}^2 + 2v_{13} - 1 - v_{12}v_{13} + v_{13},\\
B' &= v_{13}^2 v_{12} - v_{13}v_{12} + v_{13} - 1 = (v_{13}-1)(v_{12}v_{13}+1).
\end{aligned}$$

Thus

$$\begin{aligned}
(-g) &= (v_{23}-1)\cdot\sigma\cdot(v_{12}v_{13}+1) + 2\sigma(v_{12}v_{13}+1) - \sigma\cdot(v_{12}-1)(v_{13}-1),\\
(-g) &= \sigma(v_{12}v_{13}v_{23} + v_{23} + v_{12} + v_{13}).
\end{aligned}$$

This proves if F' is the function of (1.4).

PROPOSITION 3.1.1: $F'(v) = 0$.

COROLLARY: *P_3^{hyp} is the closed subscheme of Q_3 given by the equation $F' = 0$.*

Proof: P_3^{hyp} is a closed subscheme of codim 1 in Q_3. F' is contained in the ideal I of $\mathcal{O}(Q_3) = \mathbb{Z}[N_3']$ of functions vanishing on P_3^{hyp}. As F' is irreducible and the height of I is 1 we get $I = F' \cdot \mathbb{Z}[N_3']$. ∎

2. Let $X = (Y, a, b)$ be a 6–marked, stable tree of projective lines consisting of two components Y_1, Y_2 and assume that $a_i \in Y_1$, $b_i \in Y_2$ for all i. Then the curve $C(X)$ associated to X is the union of two projective lines.

X is hyperelliptic if there is an isomorphism $\sigma : Y_1 \longrightarrow Y_2$ such that

$$\sigma(Y_1 \cap Y_2) = Y_1 \cap Y_2, \quad \sigma(a_i) = b_i \quad \text{for all } i.$$

Let z_i be a global coordinate on Y_i such that $\sigma^\star z_2 = z_1$ and $z_i \cdot (Y_1 \cap Y_2) = \infty$. Then $x_i := z_1(a_i) = z_2(b_i) =: y_i$.

Let $u = (u_1, u_2, u_3) \in E_X^{\text{sym}}(k) := \{u \in E_X(k) : \text{per } u \text{ is symmetric }\}$, see [G], §1. Then $v := \text{per } u$ is a totally degenerate, symmetric matrix over k and

$$v_{ij} = t(x_j - x_i)(y_j - y_i) = t(x_j - x_i)^2$$

for some $t \in k^\star$.

Let

$$F'_{(2)} = q_{12}^2 + q_{13}^2 + q_{23}^2 - 2q_{12}q_{13} - 2q_{12}q_{23} - 2q_{13}q_{23}$$

be the homogeneous part of F' of degree 2 relative to q_{ij}.

PROPOSITION 3.2.1: $F'_{(2)}(v) = 0$

Proof: Without loss of generality one may assume that $x_1 = 0,\ x_2 = 1,\ t = 1$. Then $v_{12} = 1,\ v_{13} = x_3^2,\ v_{23} = (x_3 - 1)^2$. Then

$$v_{12}^2 + v_{13}^2 + v_{23}^2 = 1 + x_3^4 + (x_3^4 - 4x_3^3 + 6x_3^2 - 4x_3 + 1)$$

and

$$\begin{aligned} v_{12}v_{13} + v_{12}v_{23} + v_{13}v_{23} &= x_3^2 + (x_3 - 1)^2 + x_3^2(x_3 - 1)^2 \\ &= x_3^2 + x_3^2 - 2x_3 + 1 + x_3^4 - 2x_3^3 + x_3^2 \\ &= 1 + x_3^4 - 2x_3^3 + 3x_3^2 \\ &= \frac{1}{2}(v_{12}^2 + v_{13}^2 + v_{23}^2). \qquad \blacksquare \end{aligned}$$

Let $\eta : N_3' \longrightarrow \mathbb{Z}$ be the linear map which sends $m = m_{12}e_1e_2 + m_{13}e_1e_3 + m_{23}e_2e_3$ onto $m_{12} + m_{13} + m_{23}$. Then $\text{conv}(F') \cap \{y \in \mathbb{R} \otimes N_3' : \eta(y) = 2\}$ is a 2-dimensional face of $\text{conv}(F')$. Then the leading term $\xi_{(-\eta)}(F')$ of F' relative to $(-\eta)$ is equal to $F'_{(2)}$. It is $\Sigma_{\eta(m)=2}\chi_m(F') \cdot q^m$

3. Let M be a free $\mathbb{Z}$-module of finite rank n. Let T be a polyhedron in $\mathbb{R} \otimes M$ whose vertices are in $\mathbb{Q} \otimes \mathrm{M}$. This means that T is the convex hull of a finite set of points in $\mathbb{Q} \otimes \mathrm{M}$.

T defines a rational finite polyhedral cone decomposition $\sum_T$ of $\mathbb{R} \otimes \check{M}$ where $\check{M}$ is the dual $\mathbb{Z}$-module of M as follows:

Let t be a vertex of T and $\sigma_t := \{\eta \in \mathbb{R} \otimes \check{M} : \eta(x) \geq \eta(t) \text{ for all } x \in T\}$. Then σ_t is an n–dimensional rational polyhedral cone in $\mathbb{R} \otimes \check{M}$ and there is a unique polyhedral cone decomposition $\sum_T$ of $\mathbb{R} \otimes \check{M}$ whose n–dimensional cones are the system $\{\sigma_t : t \text{ vertex of } T\}$.

Let now $M = N_3'$ and $\sum = \sum_{\mathrm{conv}(F')}$. Then the torus embedding Q_Σ of Q_3 induced by $\sum$ allows a $2 : 1$ covering α onto $\mathbb{P}_3$. One can show that the period map $\mathrm{per}_3 : B_6^{\mathrm{hyp}} \cap B_6^* \longrightarrow Q_3$ can be extended uniquely to an extended period map $\mathrm{per}_3 : B_6^{\mathrm{hyp}} \longrightarrow Q_\Sigma$. The image of $\alpha \circ \mathrm{per}_3$ in $\mathbb{P}_3$ seems to be a hypersurface of $\mathbb{P}_3$ of degree 4.

4. Periods of Totally Degenerate Curves of Genus 4

1. Let $X = (\mathbb{P}_1 \times k, a, b)$ be a 8–marked, stable projective line over k. We use the notation of (2.6). Thus $v = (v_{ij}) = \mathrm{per}_4 X$ and

$$t_{ij} = \frac{x_i}{y_j} + \frac{x_j}{y_i}$$

for $2 \leq i < j$.

PROPOSITION 4.1.1:

$$t_{23}t_{24}t_{34} + 4v_{12}v_{13}v_{14} = v_{12}t_{34}^2 + v_{13}t_{24}^2 + v_{14}t_{23}^2.$$

Proof: One has

$$\begin{aligned} t_{23} &= v_{13}\frac{1}{x_3} + v_{12}x_3 \qquad \text{as } x_2 = 1, \\ t_{24} &= v_{14}\frac{1}{x_4} + v_{12}x_4, \\ t_{34} &= v_{14}\frac{x_3}{x_4} + v_{13}\frac{x_4}{x_3}, \end{aligned}$$

and thus

$$\begin{aligned} t_{23}t_{24} &= v_{13}v_{14}\frac{1}{x_3x_4} + v_{12}v_{13}\frac{x_4}{x_3} + v_{12}v_{14}\frac{x_3}{x_4} + v_{12}^2x_3x_4, \\ t_{23}t_{24}t_{34} &= v_{13}v_{14}^2\frac{1}{x_4^2} + v_{12}v_{13}v_{14} + v_{12}v_{14}^2\frac{x_3^2}{x_4^2} + v_{12}^2v_{14}x_3^2 + v_{13}^2v_{14}\frac{1}{x_3^2} + v_{12}v_{13}^2\frac{x_4^2}{x_3^2} \\ &\quad + v_{12}v_{13}v_{14} + v_{12}^2v_{13}x_4^2, \end{aligned}$$

while

$$v_{12}t_{34}^2 = v_{12}v_{14}^2\frac{x_3^2}{x_4^2} + v_{12}v_{13}^2\frac{x_4^2}{x_3^2} + 2v_{12}v_{13}v_{14}.$$

Thus

$$t_{23}t_{24}t_{34} - v_{12}t_{34}^2 = v_{13}v_{14}^2\frac{1}{x_4^2} + v_{13}^2v_{14}\frac{1}{x_3^2} + v_{12}^2v_{14}x_3^2 + v_{12}^2v_{13}x_4^2.$$

But also

$$v_{14}t_{23}^2 = v_{13}^2v_{14}\frac{1}{x_3^2} + v_{12}^2v_{14}x_3^2 + 2v_{12}v_{13}v_{14},$$
$$v_{13}t_{24}^2 = v_{13}v_{14}^2\frac{1}{x_4^2} + v_{12}^2v_{13}x_4^2 + 2v_{12}v_{13}v_{14},$$

from which follows

$$t_{23}t_{24}t_{34} - v_{12}t_{34}^2 = v_{14}t_{23}^2 + v_{13}t_{24}^2 - 4v_{12}v_{13}v_{14}. \qquad ∎$$

Let $s_{ij} := (v_{ij}-1)t_{ij}$ and $\mu := (v_{23}-1)(v_{24}-1)(v_{34}-1)$; the equation above then takes the form

$$\begin{aligned}\mu \cdot s_{23}s_{24}s_{34} + 4\mu^2 v_{12}v_{13}v_{14} = {} & v_{12}(v_{23}-1)^2(v_{24}-1)^2 s_{34}^2 \\ & + v_{13}(v_{23}-1)^2(v_{34}-1)^2 s_{24}^2 \\ & + v_{14}(v_{24}-1)^2(v_{34}-1)^2 s_{23}^2.\end{aligned}$$

Let $S_{ij} := (q_{1i}q_{1j}+1)q_{ij} - (q_{1i}+q_{1j}) \in \mathcal{O}(Q_4) = \mathbb{Z}[N_4']$ for $2 \le i < j \le 4$ and $M := (q_{23}-1)(q_{24}-1)(q_{34}-1)$. Let

$$\begin{aligned} A &:= M \cdot S_{23} \cdot S_{24} \cdot S_{34}, \\ B_{23} &:= q_{14}(q_{24}-1)^2(q_{34}-1)^2 S_{23}^2, \\ B_{24} &:= q_{13}(q_{23}-1)^2(q_{34}-1)^2 S_{24}^2, \\ B_{34} &:= q_{12}(q_{23}-1)^2(q_{24}-1)^2 S_{34}^2, \\ C &:= 4q_{12}q_{13}q_{14}M^2, \\ B &:= B_{23} + B_{24} + B_{34}, \\ \tilde{F} &:= A + C - B. \end{aligned}$$

COROLLARY 4.1.2: *$\tilde{F}$ vanishes on $P_4 \subset Q_4$.*

Proof: Let B_8^{irr} be the open subscheme of B_8 consisting of the 8–marked stable projective lines. Then B_8^{irr} is contained and dense in $B_8^\star$. If X is a point in B_8^{irr}, then $\tilde{F}(\mathrm{per}_4 X) = 0$ as $q_{ij}(\mathrm{per}_4 X) = v_{ij}$. As $\mathrm{per}_4(B_8^{\mathrm{irr}})$ is dense in P_4, the statement follows.

2. Let $\eta_{ij} : N_4' \longrightarrow \mathbb{Z}, i < j$, be the linear form which sends $e_k e_l$, $k < l$, onto

$$\begin{cases} 1 & : i = k,\ j = l, \\ 0 & : \text{otherwise.} \end{cases}$$

Let

$$\tilde{W} := \left\{ y \in \mathbb{R} \otimes N_4' : 0 \leq \eta_{ij}(y) \leq 2, \sum_{1 \leq i < j \leq 4} \eta_{ij}(y) \geq 3 \right\} \quad \text{and} \quad e = \sum_{1 \leq i < j \leq 4} e_{ij}.$$

PROPOSITION 4.2.1:

$$\operatorname{supp} \tilde{F} \subset \tilde{W},$$
$$\chi_{2e}(\tilde{F}) = 1.$$

Proof:

(1) $\operatorname{supp} A \subset \tilde{W}$, $\operatorname{supp} B_{ij} \subset \tilde{W}$, $\operatorname{supp} C \subset \tilde{W}$ and thus $\operatorname{supp}(A + C - B) \subseteq \tilde{W}$.

(2) $\chi_{2e}(A) = 1$, $\chi_{2e}(B_{ij}) = \chi_{2e}(C) = 0$ and thus

$$\chi_{2e}(\tilde{F}) = \chi_{2e}(A) + \chi_{2e}(C) - \chi_{2e}(B) = 1. \qquad \blacksquare$$

PROPOSITION 4.2.2: *$\tilde{F}$ is irreducible in $\mathcal{O}(Q_4)$ and generates the ideal in $\mathcal{O}(Q_4)$ of functions vanishing on P_4.*

Proof:

(1) Let I be the ideal in $\mathcal{O}(Q_4)$ of functions vanishing on P_4. As P_4 is irreducible of codim 1 in Q_4, the ideal I is a prime ideal of height 1 and is thus a principal ideal: $I = f \cdot \mathcal{O}(Q_4)$, $f \in \mathcal{O}(Q_4)$. $\tilde{F} \in I$ and thus $\tilde{F}$ generates I if $\tilde{F}$ is irreducible.

(2) Assume that $\tilde{F}$ is reducible. Let

$$N^+ := \{m \in N_4' : \eta_{ij}(m) \geq 0\} \quad \text{and} \quad R := \{h \in \mathbb{Z}[N_4'] : \operatorname{supp} h \subseteq N^+\}.$$

Then R is the ring of polynomials in the variables q_{ij}.

If F' is reducible in $\mathbb{Z}[N_4']$, it is also reducible in R and we may choose a generator f of I in R.

Then there is ij such that $\deg_{\eta_{ij}} f = 1$. As I is Γ_4–invariant we may assume that $ij = 34$.

Let M be the submodule of N_4' generated by e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4. There is a canonical projection $p : Q_4 \longrightarrow \operatorname{Spec}\mathbb{Z}[M]$ induced by the inclusion $M \subset N_4'$. In Lemma (5.2.2) it is shown that p gives generically a $2:1$ map $P_4 \longrightarrow \operatorname{Spec}\mathbb{Z}[M]$. This is a contradiction to the fact that $\deg_{\eta_{34}} f = 1$.

Thus $\tilde{F}$ is irreducible. $\blacksquare$

3. Let F be the function defined in (1.5).

PROPOSITION 4.3.1: $F = \tilde{F}$.

Proof:

(1) Let $\tilde{F}|_{q_{23}=1}$ be the function obtained from $\tilde{F}$ by substituting 1 for q_{23}. We claim:

$$\tilde{F}|_{q_{23}=1} = -q_{14}(q_{24}-1)^2(q_{34}-1)^2(q_{12}-1)^2(q_{13}-1)^2.$$

This is true as $C|_{q_{23}=1} \equiv 0$, $B_{24}|_{q_{23}=1} \equiv 0$, $B_{34}|_{q_{23}=1} \equiv 0$, $A|_{q_{23}=1} \equiv 0$ and $B_{23}|_{q_{23}=1} = q_{14}(q_{24}-1)^2(q_{34}-1)^2 \cdot S_{23}^2|_{q_{23}=1}$ and

$$S_{23} = ((q_{23}-1)+1)(q_{12}q_{13}+1)-(q_{12}+q_{13}) = (q_{23}-1)(q_{12}q_{13}+1)+(q_{12}-1)(q_{13}-1)$$

and $S_{23}|_{q_{23}=1} = (q_{12}-1)(q_{13}-1)$. It follows that $\tilde{F}|_{q_{23}=1} \equiv -G|_{q_{23}=1}, G$ as in (1.5).

(2) Let V be the closed subscheme of Q_4 given by the equation

$$\Delta = 0, \quad \Delta := \prod_{i<j}(q_{ij}-1).$$

Then $(\tilde{F}+G)|V \equiv 0$.

This is true because $(\tilde{F}+G)|_{q_{ij}=1} \equiv 0$ for all i,j as $\tilde{F}, G$ are invariant under the action of the permutation group $\tilde{\Gamma}_4$. It follows that $\tilde{F}+G = \Delta \cdot \tilde{H}$ for some $\tilde{H} \in \mathbb{Z}[N_4']$.

(3) $\tilde{H}$ generates a Γ_4–invariant ideal of $\mathcal{O}(Q_4)$ and for any $m \in$ supp $\tilde{H}$ one has $\eta_{ij}(m) \in \{0,1\}$. Thus

$$\tilde{H} = f_0 + f_1 q_{14} + f_2 q_{24} + f_3 q_{34} + f_{12} q_{14} q_{24} + f_{13} q_{14} q_{34} + f_{23} q_{24} q_{34} + f_{123} q_{14} q_{24} q_{34}$$

with $f_v \in \mathbb{Z}[q_{12}, q_{13}, q_{23}]$.

If $\tilde{H}(1/q_{14}, 1/q_{24}, 1/q_{34})$ denotes the function obtained from $\tilde{H}$ by replacing q_{i4} by q_{i4}^{-1} we get

$$\tilde{H}\left(\frac{1}{q_{14}}, \frac{1}{q_{24}}, \frac{1}{q_{34}}\right) = (-q_{14}q_{24}q_{34})^{-1} \cdot \tilde{H},$$

from which one gets $f_0 = -f_{123}$, $f_1 = -f_{23}$, $f_2 = -f_{13}$, $f_3 = -f_{12}$. As $f_{123} = q_{12}q_{13}q_{23}$ and supp$\Delta\tilde{H} \subset \tilde{W}$ one gets $\tilde{H} = H$. ∎

5. Curves of Genus ≥ 5

1. Let $g \geq 4$ and $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$ be a quadruple of distinct indices in $\hat{g} = \{1, \ldots, g\}$. Then ν determines a $\mathbb{Z}$– linear map $\nu' : N_4' \longrightarrow N_g'$ which maps $e_i \cdot e_j$ of N_4' onto $e_{\nu_i} e_{\nu_j}$ in N_g'. Then ν' induces a ring homomorphism $\nu' : \mathbb{Z}[N_4'] \longrightarrow \mathbb{Z}[N_g']$ and a morphism $Q_g \longrightarrow Q_4$ which will be denoted by ν_Q. One has the property:

$$\nu_Q(P_g) \subseteq P_4$$

for any ν.

Let $F_\nu := \nu'(F) \in \mathbb{Z}[N_g'] = \mathcal{O}(Q_g)$. Then $F_\nu | P_g \equiv 0$ for any ν. Let $Q_g^\star = Q_g - \{\Delta_g = 0\}$ where

$$\Delta_g := \prod_{1 \leq i < j \leq g} (1 - q_{ij}).$$

A k–valued point $v = (v_{ij})$ of Q_g belongs to $Q_g^\star$ if and only if $v_{ij} \neq 1$ for all $i < j$. Let I be the ideal in $\mathcal{O}(Q_g^\star)$ generated by $\{F_\nu : \text{ all } \nu\}$.

PROPOSITION 5.1.1: *The set of zeroes of I in $Q_g^\star$ coincides with $Q_g^\star \cap P_g$.*

The proof will be given in (5.3) below.

2. Denote by B_{2g}^{irr} the open subscheme of B_{2g} consisting of those trees of projective lines with only one component.

PROPOSITION 5.2.1: *Let X, X' be k–valued points of B_{2g}^{irr} such that* per $X =$ per X'. *Then either $X' = X$ or $X' = X^\epsilon$, $\epsilon := \epsilon_1 \cdot \ldots \cdot \epsilon_g$.*

The proof will be given at the end of section (5.2) below.

Let $X = (\mathbb{P}_1 \times k,\ a, b)$, $X' = (\mathbb{P}_1 \times k,\ a', b')$ and let z be a global coordinate on $\mathbb{P}_1 \times k$ such that $x_i = z(a_i)$, $y_i = z(v_i)$, $x_i' = z(a_i'), y_i' = z(b_i')$. We can choose z such that $x_1 = x_1' = 0$, $y_1 = y_1' = \infty$, $x_2 = x_2' = 1$. Then X is isomorphic to X' iff $x_i = x_i'$, $y_i = y_i'$ for all i.

Let $v =$ per X, $v' =$ per X'.

LEMMA 5.2.2: *Assume that*

$$v_{1i} = v_{1i}',$$

$$v_{2i} = v_{2i}',$$

for all i. Then $y_2 = y_2'$ and for all $i \geq 3$ one gets:

$$x_i' = x_i \quad \text{and} \quad y_i' = y_i$$

or

$$x_i' = \frac{v_{1i}}{v_{12}}\frac{1}{x_i} \quad \text{and} \quad y_i' = \frac{v_{1i}}{v_{12}}\frac{1}{y_i}.$$

Proof: If $v_{1i} = v_{1i}'$ for $i \geq 2$, there is $\lambda_i \in k^*$ such that

$$x_i' = \lambda_i x_i, \ y_i' = \lambda_i y_i$$

because $v_{1i}' = x_i'/y_i'$ and $v_{1i} = x_i/y_i$. Let

$$t_{ij} = \frac{x_i}{y_j} + \frac{x_j}{y_i}, \quad t_{ij}' = \frac{x_i'}{y_j'} + \frac{x_j'}{y_i'}$$

as in (2.6). Then

$$t_{ij} = \frac{(v_{1i}v_{1j}+1)v_{ij} - (v_{1i}+v_{1j})}{(v_{ij}-1)}$$

and similarly for t_{ij}'. Thus one gets $t_{2i} = t_{2i}'$ for all $i \geq 3$ from $v_{2i} = v_{2i}'$, $v_{1i} = v_{1i}'$. Now

$$\begin{aligned} t_{ij} &= v_{1j}\frac{x_i}{x_j} + v_{1i}\frac{x_j}{x_i}, \\ t_{ij}' &= v_{1j}\frac{x_i'}{x_j'} + v_{1i}\frac{x_j'}{x_i'} \\ &= v_{1j}\frac{\lambda_i x_i}{\lambda_j x_j} + v_{1i}\frac{\lambda_j x_j}{\lambda_i x_i}, \end{aligned}$$

and one gets for all $i \geq 2$:

$$v_{1i}\frac{x_2}{x_i} + v_{12}\frac{x_i}{x_2} = v_{1i}\frac{\lambda_2}{\lambda_i}\frac{x_2}{x_i} + v_{12}\frac{\lambda_i}{\lambda_2}\frac{x_i}{x_2}.$$

As $x_2 = x_2'$ one gets from $v_{12} = v_{12}'$ that $y_2 = y_2'$ and $\lambda_2 = 1$. Thus for $i \geq 3$:

$$v_{1i}\frac{1}{x_i} + v_{12}x_i = v_{1i}\frac{1}{\lambda_i x_i} + v_{12}\lambda_i x_i.$$

Solving this quadratic equation for λ_i gives two solutions:

$$\lambda_i = 1 \quad \text{or} \quad \lambda_i = \frac{v_{1i}}{v_{12}}\frac{1}{x_i^2} = \frac{y_2}{x_i y_i}.$$

Thus $x_i' = x_i$ and $y_i' = y_i$ or

$$x_i' = \frac{v_{1i}}{v_{12}}\frac{1}{x_i} = \frac{x_i \cdot y_2}{y_i x_2} \cdot \frac{1}{x_i} = \frac{y_2}{y_i}$$

and

$$y_i' = \frac{v_{1i}}{v_{12}}\frac{1}{x_i^2} \cdot y_i = \frac{y_2}{x_i}. \qquad \blacksquare$$

Remark: If X is hyperelliptic, then $X = X'$ whenever the assumptions of (5.2.2) are fulfilled.

LEMMA 5.2.3: *Assume that* $v_{1i} = v'_{1i}$, $v_{2i} = v'_{2i}$, $v_{3i} = v'_{3i}$ *for all* i. *If*

$$X' \neq X \quad \text{and} \quad X' \neq X^{\epsilon}$$

then $x_3 y_3 = y_2$.

Proof:

(1) $v_{3i} = v'_{3i}$ iff $t_{3i} = t'_{3i}$ and

$$t_{3i} = v_{1i}\frac{x_3}{x_i} + v_{13}\frac{x_i}{x_3},$$
$$t'_{3i} = v_{1i}\frac{x'_3}{x'_i} + v_{13}\frac{x'_i}{x'_3},$$
$$t'_{3i} = v_{1i}\frac{\lambda_3}{\lambda_i}\frac{x_3}{x_i} + \frac{\lambda_i x_i}{\lambda_3 x_3}.$$

The equality $t_{3i} = t'_{3i}$ is a quadratic equation for λ_i/λ_3 which has the solutions

$$\frac{\lambda_i}{\lambda_3} = 1 \quad \text{or} \quad \frac{\lambda_i}{\lambda_3} = \frac{v_{1i}\cdot x_3^2}{v_{13}\cdot x_i^2} = \frac{x_3 y_3}{x_i y_i}.$$

(2) Let $X'' = X^{\epsilon}$. Then X'' is isomorphic to

$$(\mathbb{P}_1 \times k, a'', b''),\ z(a''_1) = z(b''_i) = y''_i,\ x''_1 = 0,\ y''_1 = \infty,\ x''_i = \frac{y_2}{y_i},\ y''_i = \frac{y_2}{x_i}.$$

As per X'' = per X one gets $\mu_i \in k^*$ such that

$$x_i = \mu_i x''_i = \mu_i \frac{y_2}{y_i},$$
$$y_i = \mu_i y''_i = \mu_i \frac{y_2}{x_i}.$$

Thus $\mu_i = x_i y_i / y_2$ for all $i \geq 3$ and

$$x'_i = \lambda_i x_i = \lambda_i \mu_i x''_i.$$

If $x_i \neq x'_i$, then $\lambda_i = y_2/x_i y_i$ and $\lambda_i \mu_i = 1$, thus $x''_i = x'_i$.

(3) Let $x'_3 = x_3$. Then there is $i \geq 3$ such that $x'_i \neq x_i$. Then

$$\lambda_i = \frac{y_2}{x_i y_i} = \frac{x_3 y_3}{x_i y_i}$$

and thus $y_2 = x_3 y_3$
(4) If $x'_3 \neq x_3$ there is an index $i \geq 3$ such that $x''_i \neq x'_i$. Then $x''_3 = x'_3$, $x''_i \neq x_i$. As in (3) thus $y''_2 = x''_3 y''_3$. As $y''_2 = y_2$, $x''_3 y''_3 = y_2^2/x_3 y_3$ one also gets $x_3 y_3 = y_2$.

Proof of Proposition (5.2.1): Assume that $X \neq X'$, $X^\epsilon \neq X'$. Then apply Lemma (5.2.3) to three indices $1, 2, j$ instead of $1, 2, 3$. One gets $x_j y_j = y_2$ thus for all j. But then $X = X^\epsilon$ and from Lemma (5.2.2) we get that $X' = X$. ∎

Remark: Proposition (5.2.1) is related to a special case of a theorem of Y. Namikawa (injectivity of Torelli map), see [N], Thm. 7. However, he has given a proof only for $g \leq 3$, see [N], p. 254.

3. Now we prove Proposition (5.1.1). Let $v \in Q_g(k), F_\nu(v) = 0$ for all ν. Let v' (resp. v'') be the $(g-1) \times (g-1)$ matrix obtained from v by deleting the last column and the last row of v (resp. the column and row to the index $(g-1)$). Then v', $v'' \in Q_{g-1}(k)$ and $F_\nu(v') = 0$, $F_\nu(v'') = 0$ for all ν

If $g = 4$ we already know from Section 4 that $v =$ per X, $X \in B^\star_{2g}(k)$. If $g \geq 5$ we proceed by induction. Thus there are X', $X'' \in B^\star_{2g-2}(k)$ such that per $X' = v'$, per $X'' = v''$. As v', $v'' \in Q^\star_{g-1}$ both X', X'' are irreducible. Let

$$X' = (\mathbb{P}_1 \times k,\ a', b'), \quad X'' = (\mathbb{P}_1 \times k,\ a'', b'')$$

and

$$Y' = (\mathbb{P}_1 \times k,\ \alpha', \beta'), \qquad Y'' = (\mathbb{P}_1 \times k,\ \alpha'', \beta'')$$

with

$$\alpha' = (a'_1, \ldots, a'_{g-2}), \qquad \beta' = (b'_1, \ldots, b'_{g-2}),$$
$$\alpha'' = (a''_1, \ldots, a''_{g-2}), \qquad \beta'' = (b''_1, \ldots, b''_{g-2}).$$

Now per Y' = per Y'' and thus $Y' = Y''$ or $Y' = (Y'')^\epsilon$. If $Y' = (Y'')^\epsilon$ we replace X'' by $(X'')^\epsilon$. Thus without loss of generality we can assume $Y' = Y''$. If then $X := (\mathbb{P}_1 \times k,\ a, b)$, $a_i := a'_i$ for $i \leq g-1$, $a_g := a''_{g-1}$, $b_i := b'_i$ for $i \leq g-1$, $b_g := b''_{g-1}$, then per $X = v$.

Remark: The set of zeroes in Q_g of all F_ν seems to be P_g always, but the proof is more involved as certain combinatorial problems arise.

One also can carry out the computation of the equations which describe the set of period matrices of "totally degenerate Prym varieties", see [B], p. 618. This will be done in a forthcoming article. The Prym period matrices in Q_5 are a hypersurface in Q_5.

References

[B] A. Beauville, *Prym varieties: A survey,* in *Theta Functions*, Bowdoin 1987, Proceedings of Symposia in Pure Mathematics, Vol. 49, Part 1, AMS.

[F] H. Farkas, *Schottky–Jung theory,* in *Theta Functions*, Bowdoin 1987, Proceedings of Symposia in Pure Mathematics, Vol. 49, Part 1, pp. 459–483.

[G] L. Gerritzen, *Theta divisors on totally degenerate curves,* Bochum, manuscript, 1990.

[GHP] L. Gerritzen, F. Herrlich and M. van der Put, *Stable n-pointed trees of projective lines,* Proc. Kon. Ned. Akad. Wetenschapen, Series A, **91** (1988), 131–163.

[vG] B. van Geemen, *Siegel modular forms vanishing on the moduli space of curves,* Invent. Math. **78** (1984), 329–349.

[H] R. Hartshorne, *Algebraic Geometry,* Springer-Verlag, Berlin, 1977.

[I] J. Igusa, *On the irreducibility of Schottky's divisor,* J. Fac. Sci. Tokyo **28** (1981), 531–545.

[MF] D. Mumford and J. Fogarty, *Geometric Invariant Theory,* Springer-Verlag, Berlin–Heidelberg–New York, 1982.

[M] D. Mumford, *Tata Lectures on Theta II,* Birkhäuser-Verlag, Boston–Basel–Stuttgart, 1984.

[N] Y. Namikawa, *On the canonical holomorphic map from the moduli space of stable curves to the Igusa monoidal transform,* Nagoya Math. J. **52** (1973), 197–259.